International Journal of Theoretical Physics, Vol. 16, No. 10 (1977), pp. 721–728

Scattering Eikonal for Two Classical Relativistic Particles

P. DONATO

Université de Provence, Centre Saint-Charles Marseille and Centre de Physique Théorique, CNRS Marseille

Received: 25 January 1977

Abstract

It is shown that the classical relativistic scattering is characterized by an eikonal function of two real variables as in the nonrelativistic theory. The most general form of this function is given.

1. *Introduction*

In the framework of symplectic mechanics the problem of relativistic scattering appears in a somewhat new way. It leads to several problems that we attempt to solve.

The symplectic formalism describes a set of nonrelativistic particles by a presymplectic 2-form σ (i.e., a closed alternating covariant two-tensor of constant rank) given on the evolution space [i.e., the space of $(\mathbf{r}_i, \mathbf{v}_i, t)$ $i = 1, \cdots, n$ positions and velocities at a given time t]. The foliation defined by the distribution $x \mapsto \ker \sigma_x$ gives the laws of motion (see Souriau, 1970; or Abraham, 1967), e.g., for a set of free particles

$$\sigma(dx)(\delta x) = \sum_{i=1}^{n} mi\,[dv_{i\alpha}(\delta r_i^\alpha - v_i^\alpha \delta t) - \delta v_{i\alpha}(dr_i^\alpha - v_i^\alpha dt)], \qquad \alpha = 1, 2, 3$$

where $dx = (d\mathbf{r}_i, d\mathbf{v}_i, dt)$, $\delta x = (\delta \mathbf{r}_i, \delta \mathbf{v}_i, \delta t)$, $i = 1, \cdots, n$ are tangent vectors at x, $dv_i^\alpha \delta r_{i\alpha}$ is the usual scalar product; m_i is the mass of the ith particle. An interaction is obtained by changing the two-form, e.g.,

$$\sigma'(dx)(\delta x) = \sigma(dx)(\delta x) + \sum_{i=1}^{n} m_i [F_{i\alpha} \delta t(dr_i^\alpha - v_i^\alpha dt) - F_{i\alpha} dt(\delta r_i^\alpha - v_i^\alpha \delta t)],$$

$$\alpha = 1, 2, 3$$

$\mathbf{F}_i(x)$ $i = 1, \cdots, n$ are the forces, since: $dx \in \ker \sigma'$ iff $0 = m_i d\mathbf{v}_i - \mathbf{F}_i(x)dt$ and $d\mathbf{r}_i - \mathbf{v}_i dt = 0$.

The main advantage of this formulation is the invariance of σ' under the Galilei group action (while the usual Lagrangians have not this invariance) and its consequences (Souriau, 1970).

This evolution space is hardly appropriate in special relativity; in fact, the expression "positions and velocities at a given time", has no intrinsic meaning, the notion of simultaneity is not invariant under change of reference frame. However, one can choose as evolution space the tangent bundle of the direct product $E \times \cdots \times E$, where E is the four-dimensional Minkowski space-time; but if no precautions are taken, a noninteraction theorem is the first trap (Currie et al., 1963; Künzle, 1975; Souriau, 1970). There exists an approach that allows us to avoid the evolution space: One may consider the quotient of the evolution space by the previously described foliation $\ker \sigma$, which is interpreted as the space of motions (or space of initial conditions) (Souriau, 1970); the projection of the presymplectic form defines a symplectic structure (a closed alternating regular covariant two-tensor) which is invariant under the action of the Galilei group. This manifold of motions and group action are intrinsic, so this method can be applied to the relativistic case by translating the Galilei-invariant description on the space of motions into a Poincaré one. Then the space of motions of a relativistic dynamical system is a symplectic manifold on which the Poincaré group acts (this is the fundamental hypothesis of the symplectic formalism). But how can we construct the space of motions withoug the laws of motions defined on the evolution space? Geometric theorems allow construction and classification of these spaces in the case of a transitive action (corresponding to the elementary particles): These spaces are defined by orbits of the standard coadjoint action of the Poincaré group. E.g., the space of motions of a free particle is described by the couples $\{M, P\}$, where M is interpreted as the angular momentum and P as the 4-momentum. For more details see Souriau (1970).

The symplectic theory of scattering follows this procedure. Consider scattering nonrelativistic particles with free asymptotic motions (precise definitions below). One can show that the correspondence "free motion IN" $\mapsto$ "free motion OUT" is a sympelctomorphism of the space of free motions, which commutes with the group action and defines a scattered motion space; the transcription to the relativistic case leads to the following hypothesis: All the relativistic scattering states with free asymptotic motions are described by a symplectomorphism of the space of free relativistic motions, commuting with the Poincaré group action, and thus conserving the total 4-momentum and angular momentum of the system (Souriau, 1970). We give the most general form of this interaction map for two spinless particles, both of equal mass, and generalize the classical eikonal function.

Asymptotic Motions. This motion suggested the main hypothesis of the existence of the interaction map (cf. also Bell and Martin, 1975).

Let V denote the evolution space of a nonrelativistic system, U its space of motions, U_0 its space of *free* motions; if $x \in U$, $\mathbf{r}_i(x, t)$, $\mathbf{v}_i(x, t)$ $i = 1, \ldots, n$ are the positions and velocities at time t for the motion x. There exists a unique free motion of the system going through $(\mathbf{r}_i(x, t), \mathbf{v}_i(x, t)\ t) \in V$. (It is a sort of tangential free motion.) We shall denote it by $F(x, t) \in U_0$.

Definitions. For all $x \in U$ for which it exists

(1) $A_{\mathrm{IN}}(x) = \lim_{t \to -\infty} F(x, t)$ is called the *"IN" asymptotic motion of x.*

(2) $A_{\mathrm{OUT}}(x) = \lim_{t \to +\infty} F(x, t)$ is called the *"OUT" asymptotic motion of x.*

(3) If it exists, the map

$$x \mapsto (A_{\mathrm{IN}}^{-1}(x)) \qquad \text{will be denoted by } S_{\mathrm{OUT}}^{\mathrm{IN}} \text{ (or just } S)$$

S is defined on an open part of U_0.

Thus the scattering states for which S exists may be defined by working on the space of free motions U_0, which is easier to work with than U. Unfortunately the Keplerian motions do not admit free asymptotic motions (distinguish trajectories and motions ! However, this theory can be applied to the short-range interactions.

2. *The Interaction Symplectomorphism*

Let U' be the space of motions of a relativistic system of two free particles, without spin, both of unit mass (this last hypothesis simplifies the calculations without qualitatively changing the results).

$x \in U'$ if and only if $x = (I_1, M_1, I_2, M_2)$, where I_1, I_2, the 4-momenta, and M_1, M_2, the Lorentz angular momenta, satisfy the equations

$$\begin{aligned} \bar{I}_1 I_1 &= \bar{I}_2 I_2 = 1 \\ M_1 &= X_1 \bar{I}_1 - I_1 \bar{X}_1 \\ M_2 &= X_2 \bar{I}_2 - I_2 X_2 \end{aligned} \tag{2.1}$$

[$I_i \in \mathbb{R}^{4\times 1}$; $M_i \in \mathbb{R}^{4\times 4}$ $\bar{M}_i = G^{-1} M_i^T G$ and $\bar{I}_i = I_i^T G$ G is the diagonal 4×4 matrix $(1, -1, -1, -1)$]

$$X_i = M_i I_i + \lambda_i I_i, \qquad \lambda_i \in \mathbb{R}, \qquad i = 1, 2 \tag{2.2}$$

X_i define the space-time trajectories.

The symplectic two-form is defined by

$$\sigma(dx)(\delta x) = d\bar{X}_1 \delta I_1 - \delta \bar{X}_1 dI_1 + d\bar{X}_2 \delta I_2 - \delta \bar{X}_2 dI_2 \tag{2.3}$$

dx and δx are tangent vectors at x. The unknown is a map $S : \Omega \to \Omega$, where Ω is an open set of $U = \{x \in U' / I_1 \neq I_2\}$. We eliminate the motions with parallel space-time trajectories for the following reason: If their distance is bigger than a given value (which can be infinite) there is no interaction; if the distance is smaller the motions cannot be free. Ω is the set of scattered motions. We know a priori the following:

S conserves $I_1 + I_2, M_1 + M_2$, the total 4-momentum and the total angular momentum (see the Introduction). (2.4)

S is a symplectomorphism (2.5)

Since computing S in the given coordinates system is hardly suitable, we shall define another one.

The space-time trajectories are described by the equations (2.2); their space-time distance is minimal when X_1 and X_2 satisfy

$$(\bar{X}_1 - \bar{X}_2)I_1 = (\bar{X}_2 - \bar{X}_2)I_2 = 0 \tag{2.6}$$

We fix X_1 and X_2 satisfying (2.6).

Definitions.

(1) If $X_1 \neq X_2$, let us define J and the impact parameter a by

$$2aJ = X_1 - X_2, \qquad a \geqslant 0, \qquad \bar{J}J = -1 \tag{2.7}$$

If $a = 0$, the vector J is not determined.

(2) Let

$$I = \lambda(I_1 + I_2), \qquad \bar{I}I = 1, \qquad \lambda \in \mathbb{R} \tag{2.8}$$

(3) The polarization of the system is (Souriau, 1970)

$$W = \overline{\mathrm{vol}\,(I_2)(I_2)(X_1 - X_2)} = *(M_1 + M_2)(I_1 + I_2)$$

where vol is the usual volume element of E_4 and $\mathrm{vol}\,(I_1)(I_2)(X_1 - X_2)$ is considered as a covector. (For a detailed definition of the x operator see Souriau, 1970, 13.93).

Let

$$L = \mu \mathrm{w}, \qquad \bar{L}L = -1, \qquad \mu \in \mathbb{R} \tag{2.9}$$

and

$$2X = X_1 + X_2 \tag{2.10}$$

If K is determined by completing the orthonormal frame $\{I, J, K, L\}$ so that $\bar{K}K = -1$ and $\bar{K}I = \bar{K}J = \bar{K}L = 0$, the expression of I_1, I_2, X_1, X_2, in this frame becomes

$$\begin{aligned} I_1 &= \mathrm{ch}\,\varphi \cdot I + \mathrm{sh}\,\varphi \cdot K \\ I_2 &= \mathrm{ch}\,\varphi \cdot I - \mathrm{sh}\,\varphi \cdot K, \qquad \varphi > 0 \\ X_1 &= X + aJ \\ X_2 &= X - aJ \end{aligned} \tag{2.11}$$

If x is a two-particle motion such that $X_1 \neq X_2$, then x is characterized by X, I, J, K, L, a, and φ. We can now state the main theorem [notations and definitions of (2.1)-(2.11)].

3. *Theorem*

Let S satisfy (2.4) and (2.5), if $x = (I_1, M_1, I_2, M_2)$ let $S(x) = (I_{1\,\text{out}}, M_{1\,\text{out}}, I_{2\,\text{out}}, M_{2\,\text{out}}) = x_{\text{out}}$, then there exists a real-valued differentiable function u of the variables $H = \operatorname{ch}\varphi$ and $C = a\operatorname{sh}\varphi$ such that
(i) If x is such that $X_1 \neq X_2$ then

$$I_{1\,\text{out}} = \frac{1}{2}\left[\left(1 + \cos\frac{\partial u}{\partial C}\right)I_1 + \left(1 - \cos\frac{\partial u}{\partial C}\right)I_2\right] + \operatorname{sh}\varphi \cdot \sin\frac{\partial u}{\partial C} \cdot J$$

$$I_{2\,\text{out}} = \frac{1}{2}\left[\left(1 - \cos\frac{\partial u}{\partial C}\right)I_1 + \left(1 + \cos\frac{\partial u}{\partial C}\right)I_2\right]\ 1 - \operatorname{sh}\varphi \cdot \sin\frac{\partial u}{\partial C} \cdot J$$

$$M_{1\,\text{out}} = \left[X + \frac{\partial u}{\partial H}I + a\left(\cos\frac{\partial u}{\partial C} \cdot J + \sin\frac{\partial u}{\partial C} \cdot K\right)\right]\bar{I}_{1\,\text{out}}$$

$$- I_{1\,\text{out}}\left[\bar{X} + \frac{\partial u}{\partial H}\bar{I} + a\left(\cos\frac{\partial u}{\partial C}\bar{J} + \sin\frac{\partial u}{\partial C} \cdot \bar{K}\right)\right]$$

$$M_{2\,\text{out}} = \left[X + \frac{\partial u}{\partial H}I - a\left(\cos\frac{\partial u}{\partial C}J + \sin\frac{\partial u}{\partial C}K\right)\right]\bar{I}_{2\,\text{out}}$$

$$- I_{2\,\text{out}}\left[\bar{X} + \frac{\partial u}{\partial H}\bar{I} - a\left(\cos\frac{\partial u}{\partial C}\bar{J} + \sin\frac{\partial u}{\partial C}\bar{K}\right)\right]$$

(ii) If x is such that $X_1 = X_2$ (i.e., $a = 0$) then

$$x_{\text{out}} = \left(I_1, M_1 + \frac{\partial u}{\partial H}(I\bar{I}_1 - I_1\bar{I}), I_2, M_2 + \frac{\partial u}{\partial H}(I\bar{I}_2 - I_2\bar{I})\right)$$

or

$$x_{\text{out}} = \left(I_2, M_2 + \frac{\partial u}{\partial H}(I\bar{I}_2 - I_2\bar{I}), I_1, M_1 + \frac{\partial u}{\partial H}(I\bar{I}_1 - I_1\bar{I})\right)$$

Proof. A motion x, such that $X_1 \neq X_2$ is characterized by X, by the frame I, J, K, L and by the scalars a and φ, which are relativistic invariants. We denote for the motion x_{out} the corresponding variables by $X_{\text{out}}, J_{\text{out}}, K_{\text{out}}, I_{\text{out}}, L_{\text{out}}, \varphi_{\text{out}}$, and a_{out}. Equation (2.4) gives

$$I_1 + I_2 = I_{1\,\text{out}} + I_{2\,\text{out}} = 2I\operatorname{ch}\varphi = 2I_{\text{out}}\operatorname{ch}\varphi_{\text{out}}$$

but

$$\bar{I}I = \bar{I}_{\text{out}}I_{\text{out}} = 1$$

so that $\mathrm{ch}\,\varphi = \mathrm{ch}\,\varphi_{out}$, φ, and φ_{out} have been chosen non-negative so that

$$\varphi = \varphi_{out} \qquad \text{and} \qquad I = I_{out} \tag{3.1}$$

(2.4) gives also

$$*(M_1 + M_2)(I_1 + I_2) = *(M_{1\,out} + M_{2\,out})(I_{1\,out} + I_{2\,out})$$

so that

$$L_{out} = L$$

Computing $M_1 + M_2$ with the expressions (2.11) leads to

$$M_1 + M_2 = \mathrm{ch}\,\varphi \cdot X\bar{I} + a\,\mathrm{sh}\,\varphi \cdot J\bar{K} - (\mathrm{ch}\,\varphi \cdot I\bar{K} + a\,\mathrm{sh}\,\varphi \cdot K\bar{J})$$

(or, respectively, $M_{1\,out} + M_{2\,out} = \cdots$) such that

$$\mathrm{ch}\,\varphi \cdot X\bar{I} + a\,\mathrm{sh}\,\varphi \cdot J\bar{K} - \mathrm{ch}\,\varphi \cdot I\bar{K} - a\,\mathrm{sh}\,\varphi \cdot K\bar{J} = \mathrm{ch}\,\varphi \cdot X_{out}\bar{I}_{out}$$
$$+ a_{out}\,\mathrm{sh}\,\varphi \cdot J_{out}\bar{K}_{out} - a_{out}\,\mathrm{sh}\,\varphi \cdot K_{out}\bar{J}_{out} \tag{3.2}$$

Applying this operator to $I = I_{out}$ and considering (2.10) and (2.11) we get

$$X - X_{out} = I(\bar{X} - \bar{X}_{out})I \tag{3.3}$$

let

$$(\bar{X} - \bar{X}_{out})I = \theta(a, \varphi) \in \mathbb{R} \tag{3.4}$$

Then we get finally

$$X_{out} = X + \theta(a, \varphi) \cdot I \tag{3.5}$$

Putting (3.5) into (3.2) we get, after some simplifications,

$$a(J\bar{K} - K\bar{J}) = a_{out}(J_{out}\bar{K}_{out} - K_{out}\bar{J}_{out}) \tag{3.6}$$

thus

$$a\,\mathrm{vol}\,(K)(J) = a_{out}\,\mathrm{vol}\,(K_{out})(J_{out})$$

and

$$a\,\overline{\mathrm{vol}\,(K)(J)(I)} = a_{out}\,\overline{\mathrm{vol}\,(K_{out})(J_{out})(I)}$$

such that

$$a \cdot L = a_{out} \cdot L_{out}, \qquad \text{but } L = L_{out}$$

therefore,

$$a = a_{out} \tag{3.7}$$

Putting this in (3.6) gives

$$J\bar{K} - K\bar{J} = J_{out}\bar{K}_{out} - K_{out}\bar{J}_{out}$$

so that $J_{out} = -J\bar{K}K_{out} + K\bar{J}K_{out} = \cos\alpha \cdot J + \sin\alpha \cdot K$. Doing the same for K_{out} gives finally

$$J_{out} = \cos\alpha \cdot J + \sin\alpha \cdot K \text{ and } K_{out} = -\sin\alpha \cdot J + \cos\alpha \cdot K \tag{3.8}$$

where $\alpha(a, \varphi)$ is a differentiable function of a and φ. The results (3.1)–(3.8) and definitions (2.7)–(2.11) give at last

$$
\begin{aligned}
I_{1\,\text{out}} &= \tfrac{1}{2}[(1+\cos\alpha)I_1 + (1-\cos\alpha)I_2] - \operatorname{sh}\varphi \cdot \sin\alpha \cdot J\\
I_{2\,\text{out}} &= \tfrac{1}{2}[(1-\cos\alpha)I_1 + (1+\cos\alpha)I_2] + \operatorname{sh}\varphi \cdot \sin\alpha \cdot J\\
M_{1\,\text{out}} &= [X + \theta I + a(\cos\alpha \cdot J - \sin\alpha \cdot K)] \cdot \bar{I}_{1\,\text{out}} - I_{1\,\text{out}}[\bar{X} + \theta\bar{I}\\
&\quad + a(\cos\alpha \cdot \bar{J} - \sin\alpha \cdot \bar{K})] \qquad (3.9)\\
M_{2\,\text{out}} &= [X + \theta I - a(\cos\alpha \cdot J - \sin\alpha \cdot K)]\bar{I}_{2\,\text{out}} - I_{2\,\text{out}}[\bar{X} + \theta\bar{I}\\
&\quad - a(\cos\alpha \cdot \bar{J} - \sin\alpha \cdot \bar{K})]
\end{aligned}
$$

In view of the hypothesis (2.5) we shall now introduce the scattering eikonal.

4. *Scattering Eikonal*

Consider the one-form on the space of motions U'

$$\bar{\omega}(dx) = \bar{X}_1 dI_1 + \bar{X}_2 dI_2 \qquad (4.1)$$

The two-form σ (2.3) is the exterior derivative of $\bar{\omega}$. Using (2.11) we can write, after some reductions,

$$\bar{\omega}(dx) = 2[\bar{X}d(\operatorname{ch}\varphi I) + a\operatorname{sh}\varphi \cdot \bar{J}dK] \qquad (4.2)$$

The image of $\bar{\omega}$ by S is

$$S^*(\bar{\omega})[dS(x)] = 2[\bar{X}d(\operatorname{ch}\varphi I) + \operatorname{sh}\varphi(\theta\, d\varphi + a\, d\alpha) + \operatorname{sh}\varphi \cdot a \cdot \bar{J}dK]$$

So that

$$S^*(\bar{\omega})[dS(x)] - \bar{\omega}(dx) = 2(\theta \operatorname{sh}\varphi \cdot d\varphi + a \operatorname{sh}\varphi \cdot d\alpha) \qquad (4.3)$$

S is a symplectomorphism if and only if the exterior derivative of (4.3) is zero, i.e., (4.3) is locally exact; thus

$$\theta \operatorname{sh}\varphi \cdot d\varphi + a\operatorname{sh}\varphi\, d\alpha - d(a\operatorname{sh}\varphi \cdot \alpha) = \theta d(\operatorname{ch}\varphi) - \alpha\, d(a\operatorname{sh}\varphi)$$

must be exact. Thus there exists locally a function u such that

$$du = \theta\, d(\operatorname{ch}\varphi) - \alpha\, d(a\operatorname{sh}\varphi) \qquad (4.4)$$

Let $H = \operatorname{ch}\varphi$ and $C = a\operatorname{sh}\varphi$, then (4.4) is equivalent to

$$\theta = \frac{\partial u}{\partial H} \quad \text{and} \quad \alpha = -\frac{\partial u}{\partial C} \qquad (4.5)$$

Putting (4.5) into (3.9) we complete the proof of the first part of the theorem.

Definition. U is called the *scattering eikonal* by analogy with the nonrelativistic case.

We have defined $S : x_{\text{in}} \mapsto x_{\text{out}}$ for motions for which $X_1 \neq X_2$, if $X_1 = X_2$ we obtain a collision motion, it is possible to extend S to the submanifold of collision motions by taking a new coordinate system. This yields the second part of the theorem (for a detailed proof see Donato, 1975).

5. *Conclusion*

The scattering problem in the framework of symplectic geometry, is reduced to the study of the scattering symplectomorphism. We have given the general form of this symplectomorphism for all scattering states with free asymptotic motions, and shown that one function, the eikonal, characterizes the whole system. This function is not determined by the theory; it must be chosen on the basis of a specific interaction, perhaps, in analogy with known nonrelativistic models. Let us remark that the Keplerian scattering cannot be treated by this theory, however (there are no free asymptotic motions). In many nonrelativistic cases, the scattering is described by a symplectomorphism related to an eikonal, e.g., for the r^{-2} attractive potential or the elastic ball collisions the eikonal is, respectively,

$$U(H, C) = \pi \cdot [c - (c^2 - 2k)^{1/2}] \quad \text{and} \quad U(H, c) = 2(2H - C^2)^{1/2} - 2c \arctan \frac{(2H - c^2)^{1/2}}{c}$$

where H is the total energy, c the length of the angular momentum, k a constant, and the balls have unit diameter.

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